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Near-Extremal Black Hole Entropy and Fluctuating 3-Branes

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Abstract

We discuss the known microscopic interpretations of the Bekenstein-Hawking entropy for configurations of intersecting M-branes. In some cases the entropy scales as that of a massless field theory on the intersection. A different situation, found for configurations which reduce to 1-charge $D = 5$ black holes or 2-charge $D = 4$ black holes, is explained by a gas of non-critical strings at their Hagedorn temperature. We further suggest that the entropy of configurations reducing to 1-charge $D = 4$ black holes is due to 3-branes moving within 5-branes.

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1. Introduction

The 11-dimensional M-theory is expected to provide a unified description of different string theories. In particular, classical solutions of 11-dimensional supergravity unify various R-R and NS-NS charged soliton solutions of type IIA $D = 10$ string theory. Supersymmetric black holes in $D < 10$ have been discussed from the 11-dimensional perspective in several recent papers [1,2,3,4,5]. The aim of such discussions is twofold: (1) to find ways of understanding the Bekenstein-Hawking entropy in statistical terms using hints about the relevant degrees of freedom provided by the representation of the black holes as intersecting M-branes; (2) to obtain information about the quantum M-theory (which virtual configurations [6] are relevant, etc.) in the process of explaining the thermodynamic properties of the classical solutions.

The knowledge of the structure of possible stable classical configurations of M-theory has recently been improved substantially through the construction of composite solutions representing intersecting M-branes [7,1,2,8,9,10]. This is an important step: just like the basic 2-brane [11,7] and 5-brane [7], these intersecting (or bound) objects seem to play a fundamental role in the theory. While the basic M-branes preserve 1/2 of the supersymmetries, the intersecting configurations can preserve only $1/2^N$ where $N = 2, 3$ or 4. Upon compactification the intersecting M-branes describe the most general black holes. Of particular interest from the point of view of entropy are the supersymmetric black holes with regular horizons in $D = 4$ or 5. The representation of black holes by compactified intersecting M-branes has the advantage of being symmetric in different charges. The counting of states directly in the M-theory should provide a unified picture of the statistical description of the R-R charged (D-brane) and the NS-NS charged (conformal sigma-model) black holes found in string theory. Furthermore, it may teach us something about the M-theory itself.

In previous work [3,5] it was indeed found that the Bekenstein-Hawking entropy gives important clues about the degrees of freedom of M-theory. One such interesting development is the claim that the excitations of a 5-brane with one compact transverse direction are strings on its $D = 6$ world volume [12]. These strings originate from 2-branes intersecting the 5-brane along one dimension, with the other wrapped around the circle. A similar phenomenon leading to strings in $D = 4$ seems to be responsible for the entropy of two 5-branes intersecting over a 3-brane [5]. As we discuss in more detail below, in this case the string comes from a 5-brane compactified on T^4. In this paper we propose something even more surprising: we argue that the entropy of a 5-brane with two compact transverse directions is explained by 3-branes moving inside it. These 3-branes are 5-branes compactified on T^2. The presence of 1-branes and 3-branes in M-theory hints at a relation with the type IIB string (or F-) theory.

2. Intersecting M-branes as black holes in various dimensions

As discussed in [4], the charged $D < 10$ black hole solutions of type II string theory compactified on a torus can be represented as low-dimensional images of non-extremal generalization of supersymmetric intersecting 2- and 5-brane configurations [1,2,3] in $D = 11$. The latter can be viewed as black anisotropic p-brane solutions of 11-dimensional supergravity parametrized by N charges, Q_i, and the 'non-extremality' parameter μ. The interpretation of charges depends on specific composition of M-branes: Q_i may be a 2-brane 'electric' charge, a 5-brane 'magnetic' charge or a momentum along compact direction, i.e. Kaluza-Klein electric charge.

Sending $\mu \to 0$ we obtain an extremal black hole with N charges. On the other hand, setting all charges to zero gives the neutral Schwarzschild solution with mass $\sim \mu$. While a composite configuration of several non-extremal M-branes is not described by a static solution, the non-extremal version of supersymmetric intersecting M-brane backgrounds has a simple form which directly generalizes the 'harmonic function product' structure of extremal solutions (in the metric one is only to include the factors of Schwarzschild function $f = 1 - \frac{2\mu}{r^{D-3}}$, where $D = 11 - p$, and to change the parameters of harmonic functions from Q_i to $\mathcal{Q}_i = \sqrt{Q_i^2 + \mu^2} - \mu$).[1] The μ-extended solution, which interpolates between the extremal and the Schwarzschild solutions, thus preserves certain simple features of the extremal solution. This suggests that μ may be viewed as a parameter of 'soft' supersymmetry breaking. The idea is then to achieve some understanding of the properties of the physically relevant Schwarzschild black hole by expanding near the supersymmetric extremal point, i.e. by doing perturbation theory in μ.

2.1. Review

Let $D = 11-p$ denote the number of non-compact dimensions of intersecting M-brane configuration with p being the total number of (compact) internal coordinates y_n. Then (some of) the relevant M-brane configurations which upon dimensional reduction represent black holes with N charges in $3 < D < 11$ are

$D = 10: \quad N = 1: \ 0\uparrow$

$D = 9: \quad N = 1: \ 2; \quad N = 2: \ 2\uparrow$

$D = 8: \quad N = 1: \ 2_1; \quad N = 2: \ 2_1\uparrow$

$D = 7: \quad N = 1: \ 2_2; \quad N = 2: \ 2\bot 2$

$D = 6: \quad N = 1: \ 2_3, \ 5; \quad N = 2: \ 5\uparrow$

$D = 5: \quad N = 1: \ 2_4, \ 5_1; \quad N = 2: \ 2\bot 5; \quad N = 3: \ 2\bot 2\bot 2, \ \ 2\bot 5\uparrow$

$D = 4: \quad N = 1: \ 2_5, \ 5_2; \quad N = 2: \ 5\bot 5; \quad N = 3: \ 2\bot 2\bot 5, \ 2\bot 5\bot 5, \ 5\bot 5\bot 5;$

$\qquad N = 4: \ 2\bot 2\bot 5\bot 5, \ 5\bot 5\bot 5\uparrow$

Here $2\bot 2$ stands for the orthogonal intersection of two 2-branes over a point, $2\bot 5$ – for the intersection of a 2-brane and a 5-brane over a 1-brane, $5\bot 5$ – for intersection of two 5-branes over a 3-brane, and $\uparrow$ denotes a momentum (boost) along a compact direction which

[1] Our notation differs slightly from that in [4]: μ is rescaled by factor of 2 and the number of charges is now denoted by N, not n.

is possible to add when there is a null isometry in the extremal limit. For completeness we have included the $D = 10$ black hole (0-brane) which does not admit an M-brane representation but may be thought of as a reduction of the boosted Schwarzschild black hole in $D = 11$. The subscripts indicate the number of compact transverse coordinates. For example, 2_2 stands for the 2-brane solution which depends only on 6 of the 8 transverse coordinates. In other words, this is a 2-brane "averaged" over two compact transverse directions. This is equivalent to a special case of $2\bot 2$ where the charge of the second 2-brane set equal to zero, i.e. $2\bot 2^{(0)}$. Similarly, $2_4 = 2\bot 5^{(0)} = 2\bot 2^{(0)}\bot 2^{(0)}$ and $5_1 = 5\bot 2^{(0)}$, $5_2 = 5\bot 5^{(0)}$, etc.

The remarkable feature of unboosted configurations is that the metric is diagonal and thus all charges are contained in the antisymmetric 3-tensor field strength. Let us note that there exist also other embeddings of $D = 4$ black holes into 11-dimensional theory which involve Kaluza-Klein monopole (6-brane in $D = 10$) [13,2].

The Einstein-frame metric of the corresponding black holes in $D = 10, ..., 4$ dimensions has the following universal form [4][2]

$$ds^2_D = h^{\frac{1}{D-2}}(r)\big[- h^{-1}(r)f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_{D-2}\big] \ , \tag{2.1}$$

$$h(r) = \prod_{i=1}^{N} H_i(r) \ , \quad f(r) = 1 - \frac{2\mu}{r^{D-3}} \ , \quad H_i(r) = 1 + \frac{\mathcal{Q}_i}{r^{D-3}} \ , \quad \mathcal{Q}_i = \sqrt{Q_i^2 + \mu^2} - \mu \ .$$

The corresponding ADM mass and the Bekenstein-Hawking entropy are given by [4]

$$M = b(\sum_{i=1}^{N} \sqrt{Q_i^2 + \mu^2} + 2\lambda\mu) \ , \tag{2.2}$$

$$S = \frac{2\pi A_9}{\kappa^2} = 2b \ \frac{\mu}{T_H} = c(2\mu)^\lambda \prod_{i=1}^{N} \sqrt{\sqrt{Q_i^2 + \mu^2} + \mu} \ , \tag{2.3}$$

$$b \equiv \frac{\omega_{D-2}}{2\kappa^2}(D-3)V_p \ , \qquad c \equiv \frac{2\pi\omega_{D-2}}{\kappa^2}V_p = \frac{4\pi b}{D-3} \ .$$

Here $V_p = L_1...L_p$ $(p = 11 - D)$ is the volume of the compact internal space, $\kappa^2 (= V_p\kappa_D^2)$ is the gravitational constant in 11 dimensions, T_H is the Hawking temperature and λ is the important scaling index [8],

$$\lambda \equiv \frac{D-2}{D-3} - \frac{N}{2} \ . \tag{2.4}$$

The expression for $S(\mu, Q_i)$ interpolates between the standard Schwarzschild entropy, $S(\mu, Q_i = 0) \sim \mu^{\frac{D-2}{D-3}}$, and the entropy of extremal black holes, $S(\mu = 0, Q_i)$.

[2] Particular cases of equivalent non-extremal black hole solutions were constructed in [14,15,16,17].

The mass and the entropy satisfy the following relation

$$\frac{\partial M}{\partial \mu} = 2b \frac{\partial \ln S}{\partial \ln \mu} \ , \quad i.e. \quad T_H \frac{\partial S}{\partial \mu} = \frac{\partial M}{\partial \mu} \ . \tag{2.5}$$

This thermodynamic equation (related to the 'first law' $T_H dS(\mu, Q_i) = dM(\mu, Q_i) - \sum_{i=1}^{N} \Phi_i dQ_i$) is valid for all μ and Q_i, ranging from the Schwarzschild ($Q_i = 0$) case all the way to the extremal limit ($\mu = 0$). This explains why the proportionality between the entropy and the area of the horizon can be assumed to be true also in the extremal limit: it certainly holds in the non-extremal (or near-extremal) case and thus should be meaningful also in the limit $\mu \to 0$.[3]

The entropy has a non-zero $\mu \to 0$ limit only if $\lambda = 0$ [8], i.e. for the black holes with $D = 5, N = 3$ and $D = 4, N = 4$ which have non-singular horizons in the extremal limit. In these cases

$$M = b \sum_{i=1}^{N} \sqrt{Q_i^2 + \mu^2} \ , \qquad S = c \prod_{i=1}^{N} \sqrt{\sqrt{Q_i^2 + \mu^2} + \mu} \ . \tag{2.6}$$

Using the $D = 11$ charge quantization conditions one can further show that, for all intersecting M-brane configurations with $D = 5, N = 3$ and $D = 4, N = 4$ one has [3]

$$S_0 = (S)_{\mu=0} = c \prod_{i=1}^{N} \sqrt{Q_i} = 2\pi \prod_{i=1}^{N} \sqrt{n_i} \ , \tag{2.7}$$

where n_i are the integer values of the quantized charges. A statistical interpretation of this entropy based on the existence of the corresponding intersecting M-brane solutions $2 \perp 5 \uparrow$ and $5 \perp 5 \perp 5 \uparrow$ was discussed in [3]. These configurations are characterized by a common intersection string on a 5-brane (i.e. $c_{eff} = 6, \ d = 6$ string) with momentum along it as one quantum number, and the effective winding number proportional to $n_1 n_2$ in the $D = 5$ case, and $n_1 n_2 n_3$ in the $D = 4$ case.

This paper is primarily devoted to a statistical interpretation of the *near-extremal* terms in S suggested by the intersecting M-brane representations.

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[3] The use of μ as a 'regularisation parameter' to define the entropy in the extremal limit resolves also the following paradox. Starting with an extremal solution parametrized by an arbitrary harmonic function, one may deform it to a multicenter configuration which has the same mass, but always zero entropy (only the single-center solution may have a non-zero entropy). Thus, even an infinitesimal transverse separation of M-branes leads to a change of entropy at no energy cost. The point, however, is that the 'separated' configuration *does not* have a stable non-extremal generalization and, thus, its entropy is not well-defined.

2.2. Sub-leading entropy in the regular cases ($\lambda = 0$)

For solutions which have regular horizons, the $D = 5, N = 3$ and $D = 4, N = 4$ cases, it is interesting to study the leading near extremal corrections to the mass and Bekenstein-Hawking entropy. As follows from (2.6),

$$M = b[\sum_{i=1}^{N} Q_i + \tfrac{1}{2}\mu^2 \sum_{i=1}^{N} Q_i^{-1} + O(\mu^4)] = M_0 + E + O(\mu^4) \ , \tag{2.8}$$

$$S = c \prod_{i=1}^{N} \sqrt{Q_i}[1 + \tfrac{1}{2}\mu \sum_{j=1}^{N} Q_j^{-1} + O(\mu^2)] = S_0 + \Delta S + O(\mu^2) \ , \tag{2.9}$$

$$\Delta S = \frac{c}{\sqrt{2b}} \left[\sum_{i=1}^{N} \prod_{j \neq i=1}^{N} Q_j \right]^{1/2} \sqrt{E} \ . \tag{2.10}$$

The dependence of ΔS on E is characteristic of a $1+1$-dimensional field theory, but the dependence of the prefactor on Q_i needs an explanation. In [17,18] an explanation was proposed based on the presence of antibranes in $D = 10$ string theory picture. The antibranes seem to play a role similar to the left-movers on a string with mostly right-moving modes. As the number of antibranes (or left-movers) is sent to zero, the configuration returns to extremality. In the special case of equal charges (the Reissner-Nordsröm background with constant scalar fields) (2.10) gives

$$\frac{\Delta S}{S_0} = N \sqrt{\frac{E}{2M_0}} \ . \tag{2.11}$$

The factor of N is explained by antibranes giving contributions equal to that of the left-movers. There exists a special case, however, where the antibranes are suppressed. For the $5 \perp 5 \perp 5 \uparrow$ configuration this happens if we choose

$$\mu \ll Q_4 \ll Q_1, Q_2, Q_3 \ , \tag{2.12}$$

where Q_4 is the Kaluza-Klein charge. Now the correction to the entropy is

$$\Delta S = \frac{c}{\sqrt{2b}} \sqrt{Q_1 Q_2 Q_3 E} \ . \tag{2.13}$$

The charges Q_i have the following values [3],

$$Q_1 = \frac{n_1}{L_6 L_7} (\frac{\kappa}{4\pi})^{2/3}, \quad Q_2 = \frac{n_2}{L_4 L_5} (\frac{\kappa}{4\pi})^{2/3}, \quad Q_3 = \frac{n_3}{L_2 L_3} (\frac{\kappa}{4\pi})^{2/3}, \tag{2.14}$$

where L_i is the length of the i-th compact dimension. The left- and right-moving momenta are given by

$$P_L = \frac{2\pi N_L}{L} \ , \qquad P_R = \frac{2\pi N_R}{L} \ . \tag{2.15}$$

Near extremality we have

$$N_R = \mathcal{N} + n \ , \qquad N_L = n \ , \qquad (n \ll \mathcal{N}) \ , \quad i.e. \;\; E = \frac{4\pi n}{L} \ . \tag{2.16}$$

In terms of the integer charges, we find

$$\Delta S = 2\pi \sqrt{n_1 n_2 n_3 n} \ . \tag{2.17}$$

As explained in section 3, this is the correct relation for a $1+1$-dimensional theory on a circle of effective length $n_1 n_2 n_3 L_1$ with central charge $c_{eff} = 6$. In fact, the general entropy formula is

$$S = 2\pi \sqrt{n_1 n_2 n_3} \left(\sqrt{N_L} + \sqrt{N_R} \right) \ . \tag{2.18}$$

In section 3 we further check the consistency of this formula.

2.3. Near-extremal entropy for $\lambda \neq 0$

As follows from (2.3),(2.4) and (2.6), setting a charge to zero (i.e. going from a configuration with N charges to one with $N-1$ charges) supplies a factor of μ in the entropy. Thus, for $\lambda \neq 0$ we find, in the near-extremal limit,

$$M = M_0 + E + O(\mu^2) \ , \qquad E = 2b\lambda\mu \ , \tag{2.19}$$

$$S = c_1 \prod_{i=1}^{N} \sqrt{Q_i} \ E^{\lambda} \ , \qquad c_1 = c(b\lambda)^{-\lambda} \ . \tag{2.20}$$

In terms of the Hawking temperature T (the leading near-extremal term in T_H in (2.3)) satisfying $dE = TdS$, as implied by (2.5), we find

$$S = c_2 \prod_{i=1}^{N} (Q_i)^{\frac{1}{2(1-\lambda)}} \ T^{\frac{\lambda}{1-\lambda}} \ , \qquad c_2 = (cb^{-\lambda})^{\frac{1}{1-\lambda}} \ . \tag{2.21}$$

Since the power of T is not positive for $\lambda > 1$, the canonical ensemble description applies only for $\lambda < 1$ while for $\lambda \geq 1$ one should use the microcanonical one.

Let us recall that the entropy of a massless ideal gas in p spatial dimensions scales as $S_p \sim E^{\frac{p}{p+1}}$. In [19,8] it was noted that there are cases where $\lambda = \frac{p}{p+1}$, so that the Bekenstein-Hawking entropy may be interpreted as being due to massless fields on the p-brane. Here is a list of such cases:

(1) $D = 9, N = 1$, i.e. $p = 2, \lambda = \frac{2}{3}$. This is the basic 2-brane of M-theory, for which $S \sim \sqrt{Q} E^{\frac{2}{3}} \sim Q^{\frac{3}{2}} T^2$ [8];

(2) $D = 7, N = 1$, i.e. $p = 4, \lambda = \frac{3}{4}$, which is represented by the $2_2 = 2 \perp 2^{(0)}$ configuration. In the type IIB theory this is represented by the self-dual 3-brane in 10 dimensions. Indeed, dimensionally reducing 2_2 to $D = 10$ along one of two compact

transverse directions we get a 2_1-brane of type IIA theory, which is T-dual to the 3-brane of type IIB.[4] Here one finds the following scaling [19], $S \sim \sqrt{Q} E^{\frac{3}{4}} \sim Q^2 T^3$.

(3) $D = 6, N = 1$, i.e. $p = 5, \lambda = \frac{5}{6}$. This is the basic 5-brane of M-theory, with the scaling $S \sim \sqrt{Q} E^{\frac{5}{6}} \sim Q^3 T^5$ [8].

The feature shared by the cases (1)–(3) is that, for the extremal solution, the coupling strength is well-behaved at the location of the brane ($x = 0$). Although the scaling of S in these cases is natural, the precise number of degrees of freedom is still awaiting a complete explanation [19,8]. For other $D > 5$ black holes (including the $D = 10$ one [20] with $\lambda = \frac{9}{14}$) even the scaling exponents defy a simple explanation in terms of a free gas of massless particles. Perhaps the interactions or the massive string modes need to be taken into account.

In this paper we concentrate on the $D = 5, 4$ black holes with fewer than the critical ($N_* = 3, 4$ respectively) number of charges. If the number of charges equals $N_* - 1$, then $\lambda = \frac{1}{2}$ which leads to

$$S(\lambda = \tfrac{1}{2}) \sim \prod_{i=1}^{N} \sqrt{Q_i} \sqrt{E} \sim \prod_{i=1}^{N} Q_i T \ , \tag{2.22}$$

i.e. the scaling behavior characteristic of a $1+1$-dimensional field theory. This is not surprising since the M-brane configurations corresponding to the $D = 5, N = 2$ and $D = 4, N = 3$ cases, $2 \perp 5$ and $5 \perp 5 \perp 5$ respectively, intersect over a common string. The entropy can then be attributed to massless modes on the string, whose effective winding number is proportional to $\prod_{i=1}^{N} Q_i$. A detailed discussion of the $5 \perp 5 \perp 5$ case is provided in section 3.1.

In the cases where the number of charges is $N_* - 2$ we get $\lambda = 1$, i.e.

$$S(\lambda = 1) = \frac{E}{T} \sim \prod_{i=1}^{N} \sqrt{Q_i} E \ , \qquad T = \text{const.} \tag{2.23}$$

The corresponding configurations are $5_1 = 5 \perp 2^{(0)}$ or 2_4 (which we discuss in section 3.1) for $D = 5$, and $5 \perp 5$ for $D = 4$. This is the same scaling as in the case of the $D = 10$ 5-brane [8,12], to which 5_1 indeed reduces upon compactification on a circle. For 5_1 the entropy is naturally explained by non-critical strings in $5+1$ dimensions [12], for 2_4 – in $2+1$ dimensions, while for $5 \perp 5$ – by non-critical strings on the $3+1$-dimensional intersection [5]. The latter explanations are discussed in section 3.1.

Finally, consider the case where the number of charges is $N_* - 3$. This is possible only for the $D = 4, N = 1$ black holes, and the two relevant M-brane configurations are 2_5 and 5_2. Here we find $\lambda = \frac{3}{2}$, i.e.

$$S(\lambda = \tfrac{3}{2}) \sim \sqrt{Q} E^{\frac{3}{2}} \sim Q^{-1} T^{-3} \ . \tag{2.24}$$

In section 3.2 we argue that this scaling is explained by dynamical *3-branes* in $5+1$ dimensions.

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[4] More generally, $2 \perp 2$ gives upon reduction to $D = 10$ $2 \perp 1$ solution of type IIA theory – fundamental string intersecting R-R 2-brane. T-duality along the string direction converts this into 3-brane of type IIB with extra momentum along one direction equal to string winding number or charge of the second 2-brane in $2 \perp 2$. Setting this charge to zero gives unboosted 3-brane, which is indeed T-dual to 2_1-brane in $D = 10$.

3. The Entropy of Intersecting 5-Branes

We shall now focus on the intersecting M-brane configurations which, upon dimensional reduction, correspond to the $D = 4$ black holes with N =3, 2, or 1 charges. For each of these cases we suggest a separate microscopic explanation of the near-extremal Bekenstein-Hawking entropy.

3.1. Entropy due to non-critical strings

Let us start with the intersecting M-branes which, upon wrapping them over a T^7, reduce to $D = 4$ black holes with 3 charges. The example that we shall discuss in detail is $5\perp5\perp5$, involving n_1 5-branes positioned in the (12345) hyperplane, n_2 5-branes positioned in the (12367) hyperplane, and n_3 5-branes positioned in the (14567) hyperplane.[5] The classical solution describing this configuration with the non-extremality parameter μ is given by [4]

$$\begin{aligned} ds_{11}^2 &= (H_1H_2H_3)^{2/3}\big[(H_1H_2H_3)^{-1}(-fdt^2+dy_1^2)+(H_1H_2)^{-1}(dy_2^2+dy_3^2) \\ &\quad +(H_1H_3)^{-1}(dy_4^2+dy_5^2)+(H_2H_3)^{-1}(dy_6^2+dy_7^2)+f^{-1}dr^2+r^2d\Omega_2^2\big]\ , \\ H_i &= 1+\frac{\mathcal{Q}_i}{r}\ , \qquad f=1-\frac{2\mu}{r}\ , \qquad \mathcal{Q}_i=\sqrt{Q_i^2+\mu^2}-\mu\ . \end{aligned} \tag{3.1}$$

The near-extremal Bekenstein-Hawking entropy of this solution is given by (2.20),(2.22)

$$S=\frac{8\pi^{3/2}}{\kappa}\left(Q_1Q_2Q_3\prod_{i=1}^{7}L_iE\right)^{1/2}=2\sqrt{\pi}\sqrt{n_1n_2n_3L_1E}\ , \tag{3.2}$$

where we have used (2.14). This has the same form as the entropy vs. energy of a $1+1$ dimensional field theory with central charge c_{eff} defined on a circle of length L_{eff},

$$S=2\sqrt{\pi}\sqrt{\tfrac{1}{6}c_{eff}L_{eff}E}\ . \tag{3.3}$$

Thus, the near-extremal Bekenstein-Hawking entropy is explained by massless modes on the intersection of the 5-branes. One possible identification is that $c_{eff}=6$, while $L_{eff}=n_1n_2n_3L_1$. This is consistent with what was necessary to explain the extremal entropy of the boosted $5\perp5\perp5$ solution [3]. Note also that (3.2) agrees with the general formula (2.18) after we substitute $E=4\pi N_L/L_1=4\pi N_R/L_1$.

In [3] it was proposed that the massless degrees of freedom come from collapsed 2-branes with triple boundaries, one boundary lying in each of the 5-brane hyperplanes. Such 2-branes provide triple connections of the 5-branes near the intersection string. An

[5] The other possibilities where some of the 5-branes are replaced by the 2-branes are related to this by the U-duality of the $D=4$ theory.

argument analogous to that in [21] explains why the effective length of circle on which such degrees of freedom move is enhanced by the factor $n_1 n_2 n_3$.

Let us now set $n_3 = 0$ (i.e. $\mathcal{Q}_3 = 0$, $H_3 = 1$) in (3.1). The resulting solution describes the $5\perp5$ configuration which, upon compactification, reduces to the $D = 4$ black hole with 2 magnetic charges, n_1 and n_2. As shown in [5], its near-extremal entropy reveals an interesting connection with non-critical strings propagating on the $3+1$-dimensional intersection. Let us repeat the essential points of the argument in [5].

As a function of the energy, $E = M - M_0$, the near-extremal entropy is (2.20),(2.23)

$$S = 4\pi\sqrt{Q_1 Q_2}\ E\ = 2^{7/6}\pi^{5/6}\kappa^{2/3}\left(\frac{n_1 n_2}{L_4 L_5 L_6 L_7}\right)^{1/2} E\ . \tag{3.4}$$

This has the same form as the entropy of a gas of strings at its Hagedorn temperature (which is dominated by the contribution of one long non-winding string)

$$S = 2\pi\sqrt{\tfrac{1}{6}c_{eff}\alpha'_{eff}}\ E\ . \tag{3.5}$$

Here c_{eff} is the central charge of the world sheet degrees of freedom, while $T_{1\ eff} = \frac{1}{2\pi\alpha'_{eff}}$ is the string tension. To achieve agreement between (3.5) and (3.4), we take [5]

$$c_{eff} = 6\ , \qquad T_{1\ eff} = \frac{L_4 L_5 L_6 L_7}{n_1 n_2}\, T_5\ , \tag{3.6}$$

where

$$T_5 = \left(\frac{\pi}{2}\right)^{1/3}\kappa^{-4/3}\ , \tag{3.7}$$

is the tension of the 5-brane of M-theory [3]. For $n_1 = n_2 = 1$ the string tension equals that of a 5-brane wrapped over the T^4 in the (4567) directions. For higher values of n_1 and n_2, the tension of the wrapped 5-brane is reduced by a multiplicative factor. A similar reduction of tension was necessary for explaining the entropy of a D-string moving within a number of parallel type IIB 5-branes [12]. The value of c_{eff} seems puzzling since, for a string in $3+1$ dimensions, one would expect $c_{eff} = 3$. However, as we will show in the next paragraph, such a naive expectation fails also for a string in $2+1$ dimensions – one needs an exact theory, such as the D-branes, to count the central charge.

As a slight digression, let us note that the 5-brane wrapped over T^4 also explains the near-extremal entropy of the 2_4 configuration, which reduces to a single-charge $D = 5$ black hole. In this case, the near-extremal entropy is related to energy according to (2.23) [4]

$$S = 2\pi\sqrt{Q}E\ , \qquad Q = \frac{n}{L_3 L_4 L_5 L_6}\left(\frac{\kappa}{\sqrt{2}\pi}\right)^{4/3}\ , \tag{3.8}$$

where $3, 4, 5$ and 6 are the compact directions transverse to the n 2-branes. Once again, the entropy has the form (3.5) characteristic of a string gas at Hagedorn temperature, with

$$c_{eff} = 6\ , \qquad T_{1\ eff} = \frac{L_3 L_4 L_5 L_6}{n}\, T_5\ . \tag{3.9}$$

Now the non-critical string originating from the wrapped 5-brane moves in 2+1 dimensions (within the 2-branes). From this point of view, the value of the central charge necessary for matching the entropy seems mysterious. However, upon reduction to 10 dimensions this value is confirmed using the D-brane count (the resulting configuration, 2_3, is T-dual to the D5-brane considered in [12]).

As we have seen, the near-extremal entropy of the $5\perp5$ configuration is reproduced by non-critical strings which may be interpreted as 5-branes wrapped around T^4. Presumably, this is related to the fact that there exists an extremal $5\perp5\perp5$ configuration which preserves 1/8 of the original supersymmetries [1,2,9]: we can add a 5-brane in the $(a4567)$ hyperplane, with $a = 1, 2$ or 3. From the point of view of the extremal $5\perp5$, this adds a long straight string on the $3 + 1$ dimensional intersection. We believe that the "almost-supersymmetric" objects responsible for the near-extremal excitations of the $5\perp5$ configuration are large loops of such string (if the string bends slowly, the supersymmetry breaking is small). Unlike the completely straight strings, such loops do not carry any charge (winding number).

Let us emphasize that c_{eff} is equal to 6 both for the $5\perp5\perp5$ intersection string and for the gas of non-critical strings on the $5\perp5$ intersection. This is necessary for consistency of our interpretation because in both cases we are dealing with the same non-critical string theory.[6] On the $5\perp5$ intersection the strings are free to wander around. Adding a large number of 5-branes in the third hyperplane pins one long string down and makes it semiclassical, so that the entropy is due to its small fluctuations.

3.2. Entropy due to fluctuating 3-branes

By analogy, we may now interpret the near-extremal entropy of the solution (3.1) with $n_2 = n_3 = 0$. This describes n_1 coincident 5-branes positioned in the (12345) hyperplane and "averaged" over the compact directions 6 and 7, i.e. 5_2. Upon compactification on T^7, it reduces to the $D = 4$ black hole with one magnetic charge, n_1. We expect that the near-extremal entropy of this solution is related to the existence of the $5\perp5$ configuration in the following sense: a 5-brane in the $(abc67)$ hyperplane wrapped around the (67) directions is a straight 3-brane positioned inside the n_1 5-branes (a, b and c are three orthogonal directions within the (12345) hyperplane). Clearly, the 3-brane has massless fermionic modes associated with the fact that it breaks 1/2 of the supersymmetries present on the 5-brane, while their bosonic partners are associated with small transverse fluctuations. We believe that the departure from extremality excites these massless modes, making the 3-brane a dynamical object moving inside the 5-branes. Thus, the near-extremal Bekenstein-Hawking entropy should coincide with that of a gas of 3-branes of tension $T_{3\ eff} \sim L_6 L_7 T_5$ moving in $5 + 1$ dimensions.

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[6] It may be related to the string theory on the 5-brane discussed in [22] and in [12].

The near-extremal entropy of (3.1) with $n_2 = n_3 = 0$ ($H_2 = H_3 = 1$) is given by (2.20),(2.24)

$$S = \frac{8\sqrt{\pi}\kappa}{3\sqrt{3\prod_{i=1}^{7} L_i}} \sqrt{Q_1}\ E^{\frac{3}{2}} \ . \tag{3.10}$$

Using (2.14) we rewrite this as

$$S = \frac{4\sqrt{\pi n_1}}{3T_{3\ eff}\sqrt{3V}}\ E^{\frac{3}{2}} \ , \tag{3.11}$$

where

$$T_{3\ eff} = L_6 L_7 T_5 \ , \qquad V = L_1 L_2 L_3 L_4 L_5 \ . \tag{3.12}$$

For this formula to be applicable, the entropy has to be macroscopic, i.e. $S \gg 1$. This puts a lower bound on E. If we assume that L_6 and L_7 are of order $l_P \sim \kappa^{2/9}$, the 11-dimensional Planck scale, then we find

$$E \gg \left(\frac{V}{n_1 l_P^8}\right)^{1/3} \ . \tag{3.13}$$

At the same time, there is also an upper bound on E coming from the fact that the 5-brane is near extremality, $E \ll M_0$, which implies

$$E \ll \frac{n_1 V}{l_P^6} \ . \tag{3.14}$$

When either n or V/l_P^5 is large, there is a large range of values of E compatible with both (3.13) and (3.14).

We note that $S(E)$ in (3.11) grows faster than for a gas of strings. For a $d+1$-dimensional free field theory, $S(E)$ grows as $E^{\frac{d}{d+1}}$ which is slower than for a gas of strings. This leads us to believe that the behavior (3.11) can be explained only by dynamical p-branes with $p > 1$. Unfortunately, there is no known exact quantum theory for such objects. The best one can do is to study thermodynamics of p-branes in the semiclassical approximation (see, for example, [23,24]), where one finds that, for sufficiently high energy, $S \sim E^{\frac{2p}{p+1}}$.

Thus, the entropy (3.11) is consistent with that of a gas of 3-branes. Since the growth of S is faster than linear, the microcanonical ensemble is dominated by a single large 3-brane.[7] The thermodynamics here is highly unusual since the temperature falls off with increasing energy as $1/\sqrt{E}$. Note, however, that to compare with the Bekenstein-Hawking entropy, (3.11), it is sufficient to work in the microcanonical ensemble, which is well-defined even for p-branes with $p > 1$.

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[7] Clearly, this maximizes the entropy for a given total energy: dividing one p-brane of energy E into two p-branes of energy $\frac{1}{2}E$ would reduce the entropy by a factor $2^{\frac{2}{p+1}}$.

For a semiclassical 3-brane, the world volume is described by a $3+1$-dimensional theory containing some number, m, of massless bosons and the same number of massless fermions. If the internal volume of the 3-brane is v, and the world volume energy is e, standard statistical mechanics gives

$$S = \tfrac{2}{3}\sqrt{\pi}(mv)^{1/4}e^{3/4} \ . \tag{3.15}$$

In the semiclassical (gaussian) approximation the connection between the target space energy, E, and the world volume energy is similar to that for strings,

$$E^2 = xe \ . \tag{3.16}$$

For strings the exact quantum theory is available, and one finds that $x \sim T_1 v$. For 3-branes (3.16) holds in the semiclassical approximation (with $x \sim v^{1/3}T_3^{1/2}$), but we do not yet know how to determine x in the exact theory.

We could turn the logic around and, assuming that the Bekenstein-Hawking entropy is explained by the fluctuating 3-branes, try to learn something about their quantum theory. First of all, (3.11) implies that the relation (3.16) holds in the full quantum theory, not just in the semiclassical approximation. Furthermore, the matching of entropy requires that

$$x \sim v^{1/3}T_{3\ eff}^{4/3}V^{2/3} \ . \tag{3.17}$$

The major difference from the case of strings is that this relation involves the target space volume. This suggests that the 3-brane theory contains strong space-time infrared effects. Perhaps such effects renormalize the 3-brane tension from its bare value, $T_{3\ eff}$.

String theory ($p=1$) seems to be the only case where such effects are not present, which is intimately related to the conformal invariance on the world sheet. Indeed, for $p=1$ neither x nor $S(E)$ depends on the target space volume, V. For the other solvable case, a gas of particles ($p=0$) in $d+1$ dimensions, $S \sim V^{\frac{1}{d+1}}E^{\frac{d}{d+1}}$. Here V enters the entropy with a positive power, and therefore it is natural that for 3-branes it enters with a negative power. It is not yet clear to us why this power is $-\frac{1}{2}$.

4. Some Remarks on 'the 3-brane within a 5-brane'

The appearance of a 3-brane in $D=11$ at the intersection of two 5-branes suggests a connection to the 3-brane of type IIB theory. This connection can be made more explicit as follows.[8] Dimensional reduction of the $5\perp5$ along one of the common 3-brane directions gives the $4\perp4$ configuration (with a common 2-brane) in the type IIA theory. T-duality

[8] Note that there is also another $D=11$ counterpart of the type IIB 3-brane. This is the 2_2 configuration, i.e. the 2-brane with two isometric transverse dimensions or, equivalently, $2\perp2$ with the second charge set equal to zero (this configuration was already mentioned in section 2.3).

relates this to a '2-brane within 6-brane' configuration, $6\|2$.[9] $4\perp4$ is also T-dual to the $5\perp3$ configuration of type IIB, i.e. to a D5-brane intersecting a D3-brane over a 2-brane. At the same time, T-duality along one of the common 2-brane directions transforms $4\perp4$ into $(3\perp3)_1$, i.e. $3\perp3$ with one compact transverse direction. Both $4\perp4$ and $(3\perp3)_1$ dimensionally reduce to the $3\perp3$ configuration of the $D=9$ theory. Setting the second charge to zero gives 3_2 in $D=9$.

In the reductions mentioned above, the intersection becomes a 2-brane, while in the original M-theory configuration the two 5-branes intersect over a 3-brane. We may ask whether it is possible to reduce to string theory in such a way that the intersection remains a 3-brane. The answer is yes, but the reduced configuration involves not only D-branes, but also NS-NS charged branes. For example, we may dimensionally reduce $5\perp5$ along one of the two compact internal dimensions of the second 5-brane which are orthogonal to the first one (y_6 or y_7 in (3.1)). This gives the $5\perp4$ solution of type IIA theory, i.e. an NS-NS 5-brane intersected by a R-R 4-brane over a 3-brane. Since the 4-brane direction orthogonal to the 5-brane, y_6, is an isometry, we may apply T-duality along this direction to obtain the $5_1\|3$ solution of type IIB theory[10], i.e. a 3-brane lying within a 5-brane, with one of the transverse dimensions being periodic.

This 5_1 configuration is the Kaluza-Klein 5-brane of type IIB theory on $M^9\times S^1$ [26] (i.e. the object magnetically charged under the Kaluza-Klein gauge field) which is a singlet under the $SL(2,R)$ symmetry. Its connection with the 5_2 background in $D=11$ may be seen as follows. We may start with (3.1) and set $H_1=H,\ H_2=H_3=f=1$. The corresponding metric and 3-form field strength are

$$ds^2_{11}=H^{2/3}\big[H^{-1}(-dt^2+dy_ndy_n)+dy_6^2+dy_7^2+dx_idx_i\big],\ \ \mathcal{F}_4=3*dH\wedge dy_6\wedge dy_7, \quad (4.1)$$

where $n=1,...,5;\ i=1,2,3;\ r^2=x_ix_i$. Compactifying to $D=10$ along y_7 we get the NS-NS 5-brane 5_1 of type IIA theory in $M^9\times S^1$ described by the $SO(3)$ symmetric string sigma-model,

$$L_{10}=-\partial t\bar\partial t+\partial y_n\bar\partial y_n+H(x)(\partial y_6\bar\partial y_6+\partial x_i\bar\partial x_i)+B_i(x)(\partial y_6\bar\partial x^i-\bar\partial y_6\partial x^i)\ , \quad (4.2)$$

where $dB=-*dH$ and the dilaton is $\phi=\frac{1}{2}\ln H$. T-duality along y_6 converts this 'H-monopole' sigma-model into the Kaluza-Klein monopole sigma-model with no dilaton and no antisymmetric 2-tensor and the following non-diagonal metric,

$$ds^2_{10}=-dt^2+dy_ndy_n+H^{-1}(x)(d\tilde y_6+B_idx^i)^2+H(x)dx_idx_i\ . \quad (4.3)$$

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[9] For a discussion of intersecting p-brane solutions in $D=10$ and T-duality see [2,25].

[10] T-duality along one of the common directions gives a D3-brane intersecting a NS-NS 5-brane over a 2-brane. T-duality along one of the 5-brane directions that does not belong to the D4-brane produces a D5-brane intersecting a NS-NS 5-brane over a 4-brane ($SL(2,R)$ transformation interchanges the two 5-branes).

This is the metric describing the Kaluza-Klein 5-brane of type IIB theory. Note that the direct lift of this metric into $D = 11$ gives a Kaluza-Klein 6-brane which becomes the type IIA 6-brane upon reduction along y_6 [13].

Dimensional reduction of $5_1\|3$ along y_6 gives the $5\|3$ configuration in 9 dimensions (upon further reduction to $D = 4$ this in turn gives a black hole (2.1) in $D = 4$ with 2 charges). Thus, in the $D = 9$ theory we find a D3-brane inside a Kaluza-Klein 5-brane. If we switch off the charge of the 3-brane, we return to the 5_1 configuration in $D = 10$, which reduces to the Kaluza-Klein 5-brane in $D = 9$. We believe that the 3-brane interpretation of the entropy of the latter can now be rephrased in terms of a fluctuating 3-brane of type IIB theory which lies within the Kaluza-Klein 5-brane. Unfortunately, since the Kaluza-Klein 5-brane is a purely NS-NS object, this situation cannot be described in D-brane terms.

Let us emphasize that the 5-branes in 11, 10 and 9 dimensions have different scaling of the near-extremal entropy. For the $D = 11$ 5-brane the scaling is that of a massless field theory; for the $D = 10$ 5-brane, which originates from the 5_1 configuration of M-theory, it is that of a string; for the $D = 9$ 5-brane, which originates from 5_2, it is that of a 3-brane. This variety of 'non-critical' p-branes in $D = 6$ may be related to type IIB compactifications on K3. At the point in moduli space where a 2-cycle degenerates a 3-brane wrapping around it gives rise to a tensionless string [27], while a 5-brane gives rise to a tensionless 3-brane. These objects may be related to the non-critical strings and 3-branes that we are finding.

The low-energy fluctuations of the M-theory 5-brane are known to be described by the $N = 2$ tensor multiplet in $D = 6$, whose bosonic fields are B^-_{mn} and 5 scalars. The scalars have the interpretation of the transverse positions of the 5-brane in $D = 11$. If one is interested in the 5_1 configuration in $D = 11$, which is characterized by non-critical strings, then one of the scalars needs to be taken compact. This was indeed the case for the quantization procedure of [22]. If we are interested in the 5_2 configuration, then two of the scalar fields need to be made compact. Perhaps such a world-volume theory has a BPS saturated soliton solution describing the embedded 3-brane. This would be similar in spirit to the construction of [28] where the 3-brane soliton of $D = 6$ supersymmetric abelian gauge theory was found.

Our eventual goal is to derive an effective action for the 3-brane in $D = 6$, similar to the κ-supersymmetric action in [28], and below we speculate on the possible form of this action. A clue may come from consideration of an effective action for the fluctuations of the $5\perp5$ configuration in 11 dimensions (with the second charge eventually set equal to zero). The first step towards constructing actions describing intersecting M-brane configurations should be to identify the corresponding zero modes (as was done for the basic 5-branes in [29]). Since the extremal $5\perp5$ solution breaks 24 out of 32 supersymmetries of 11-dimensional theory we expect to find 24 (Goldstone) fermionic zero modes and hence 12 bosonic zero modes. The 'anisotropic 7-brane' form of the extremal $5\perp5$ metric ((3.1) with $H_3 = 1, f = 1$) implies the existence of four normalizable translational zero modes, i.e. collective coordinates $X^i(\xi)$ $(i = 1, ..., 4)$. In addition, there should be 8 bosonic

zero modes $Z^p(\xi)$ $(p = 1, ..., 8)$ coming from the antisymmetric 3-tensor (and related to breaking of the corresponding gauge symmetry). The absence of the full Lorentz symmetry of the 7-brane implies that the resulting low-energy static-gauge action will have only the reduced $SO(1,3) \times SO(2) \times SO(2)$ global symmetry

$$S_{5\perp 5} \sim \int d^8\xi (\eta^{\mu\nu}\partial_\mu X^i \partial_\nu X^i + c_1 \partial_a X^i \partial_a X^i + c_2 \partial_s X^i \partial_s X^i + ...) \ . \tag{4.4}$$

Here $\mu, \nu = 0, 1, 2, 3;\ a = 4, 5;\ s = 6, 7$. The values of the 'anisotropy constants' c_k depend on the values of the two 5-brane charges Q_k ($c_1 = c_2$ if $Q_1 = Q_2$, $c_{1,2} = 1$ if $Q_{2,1} = 0$). The 5_2-brane action should be a result of certain ('double-dimensional') reduction of (4.4) (with $c_2 = 1$) along ξ^6, ξ^7.

To determine the field content of the action for the dynamical 3-brane moving inside the 5-brane it is useful to use supersymmetry considerations. The 5-brane theory has 16 conserved supercharges, and the presence of a 3-brane breaks half of them. Thus, the residual supersymmetry of the 3-brane effective action is $N = 2$ in $D = 4$. This action must contain 8 Goldstone fermions, which naturally combine into two $D = 4$ Majorana fermions. By supersymmetry, the action should also contain 4 massless bosonic degrees of freedom. We know that two of these bosons are the scalars describing the transverse fluctuations of the 3-brane in $D = 6$. One possible choice of the $N = 2$ multiplet containing the required degrees of freedom is the vector multiplet. Then the low-energy effective action for a single 3-brane is described by the $N = 2$ supersymmetric $U(1)$ theory (one may then speculate that multiple 3-branes are described by the non-abelian $N = 2$ vector multiplet). Another possibility is to use the $N = 2$ hypermultiplet. The two extra scalars may then be interpreted as corresponding to the two compact coordinates transverse to the 5-brane.[11] It would be interesting to study the 3-brane dynamics in $D = 6$ in more detail.

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[11] Note also that the number of on-shell degrees of freedom, $4 + 4$, is the same as in the κ-supersymmetric action of a 3-brane moving in 8 space-time dimensions [30].

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